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PROCESS ALGEBRA SEMANTICS FOR QUEUES

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Process algebra semantics for queues*)

by

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ABSTRACT

An unbounded queue over a finite set of data values is modeled as a state transition system. After behavioural abstraction its behaviour is a process Q in A^{∞} where A is the collection of the input and output actions for the queue. A specification of Q by means of recursion equations is provided, using a new auxiliary operator on processes. It is shown that this operator is necessary in the sense that it is not possible to specify Q using recursion equations built from sequential, alternative and parallel composition only. Sequential composition of two queues is shown to realise another queue.

KEY WORDS & PHRASES: *Process algebra, queue, fixed point equations, bisimulation*

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*) This report will be submitted for publication elsewhere

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INTRODUCTION

An *unbounded queue*, (or buffer working in FIFO mode) is a device able to sequentially receive data values from a domain D, to store them and to deliver them in the order in which they were received.

In order to describe the behaviour of queue Q it is assumed that D is finite, moreover the input actions and output actions together form an alphabet of actions for Q. These actions exclude one another in time. In particular, for each $d \in D$ there are these actions:

d : input d

$\underline{d}$: output d

Sets $\underline{D}$ and A are defined by $\underline{D} = \{\underline{d} \mid d \in D\}$, $A = D \cup \underline{D}$. Then Q can be semantically described as a state transition system over A.

After behavioural abstraction a process Q in A^{∞} is obtained. A^{∞} is the projective limit model of processes introduced in BERGSTRA & KLOP [3].

In fact, behavioural abstraction yields a process $\pi(s)$ for each state s of the transition system for Q. A state s is characterised by the sequence $s \in D^*$ of data that can be output before new input is received. Within A^{∞} there are identities that relate the various $\pi(s)$ to one another: Amongst these identities there is an elegant but infinite subset which completely describes all $\pi(s)$:

$$\begin{cases} \pi(\emptyset) = \sum_{d \in D} d.\pi(d), \text{ and for all } s: \\ \pi(s*d) = \sum_{e \in D} e.\pi(e*s*d) + \underline{d}.\pi(s). \end{cases}$$

Assuming that Q is initially empty its behaviour is given by

$$\pi(\emptyset) = Q$$

Working in the two sorted system, containing both A^{∞} and the state transition system as sorts as well as the auxiliary operator π, the above equations provide a finite equational specification of Q.

The main problem addressed here is how to specify Q within A^{∞} without the use of an extra state transition system. Besides it is shown that the sequential composition of two queues again yields a queue. This involves ACP, algebra of communicating processes from [3], the method of [4] to combine two queues into a network and the abstraction mechanism from [5].

The results of this paper are these.

(1) There are auxiliary operators $/\!\!\backslash\underline{d}$, $\triangle\underline{d}$ for $d \in D$, on A^{∞} which can be specified by means of finitely many recursion equations which allow to specify Q by means of a single recursion equation over

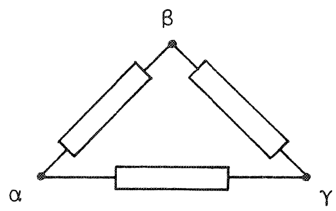
$$A^{\infty}(+, \cdot, /\!\!\backslash\underline{d}, \triangle\underline{d}).$$

(2) Q cannot be defined using a finite system of guarded fixed point equations over

$$A^{\infty}(+, \cdot, ||, |\!\lfloor).$$

(3) Let α,β,γ be three locations, and let $Q^{\alpha,\beta}$, $Q^{\beta,\gamma}$, $Q^{\alpha,\gamma}$ be queues which take inputs in α,β,α and produce outputs in β,γ,γ respectively.
Now sequentially composing $Q^{\alpha,\beta}$ and $Q^{\beta,\gamma}$ produces $Q^{\alpha,\gamma}$.
In ACP terms:

$$Q^{\alpha,\gamma} \underline{\leftrightarrow}_{I} \partial_{H}(Q^{\alpha,\beta} \,||\, Q^{\beta,\gamma})$$



1. PROCESS ALGEBRA

Let A be a finite set of atomic actions. Processes are configurations of actions of A. Composition tools for processes are:

+ alternative composition
. sequential composition
|| parallel composition (merge)
$|\!\lfloor$ left merge.

The axioms of PA below, taken from [1] describe the operators; a varies over A.

$$X + Y = Y + X$$
$$X + (Y+Z) = (X+Y) + Z$$
$$X + X = X$$
$$(X+Y).Z = X.Z + Y.Z$$
$$(X.Y).Z = X.(Y.Z)$$
$$X \| Y = X \lfloor\!\lfloor Y + Y \lfloor\!\lfloor X$$
$$a \lfloor\!\lfloor X = a.X$$
$$(a.X) \lfloor\!\lfloor Y = a(X \| Y)$$
$$(X+Y) \lfloor\!\lfloor Z = X \lfloor\!\lfloor Z + Y \lfloor\!\lfloor Z \,.$$

Because A is finite PA is finite too. As an equational specification it has an initial algebra, called A_ω.

For each n one may identify processes which differ at depth n only, thus obtaining a congruence $\equiv_n$ on A_ω. $A_n = A_\omega/\equiv_n$ is a model of PA as well. The structures A_n ($n \in \omega$) have a projective limit A^∞ which contains A_ω as a proper substructure. (A^∞ was introduced in [3] and is in fact an algebraic reconstruction of the topological process semantics in DE BAKKER & ZUCKER [1,2]). A^∞ serves us as a standard model for processes.

For processes $X \in A_\omega$ one defines projections $(x)_n$ as follows:

$$(a)_n = a$$
$$(aX)_1 = a$$
$$(aX)_{n+1} = a(X)_n$$
$$(X+Y)_n = (X)_n + (Y)_n.$$

The congruence $\equiv_n$ can be formally defined by $X \equiv_n Y \Longleftrightarrow (X)_n = (Y)_n$.
An element of A^∞ is just a sequence

$$(P_1, P_2, P_3, \ldots)$$

with $P_n \in A_n$ (i.e. $(P_n)_n = P_n$) and for all n: $(P_{n+1})_n = P_n$. The operations $+, \cdot, \|$ and $\lfloor\!\lfloor$ are defined componentwise.

REMARK. PA does not describe composition with communication. In the framework of ACP (see [4,5] for an introduction), communication requires the following extra features.

(i) a larger set of atoms, say B such that $B \supseteq A$.

(ii) a constant δ, for *deadlock* in $B - A$

(iii) a *communication function* $\cdot|\cdot: B \times B \to B$ which is commutative and associative and satisfies $\delta|b = \delta$, and $a|b = \delta$ (for $a \in A$).

(iv) a subset H of $B - A$ of *subatomic actions*. Usually $b \in H \iff \exists b' \in B$ $b|b' \neq \delta$.

The axioms of ACP, (not repeated here) will define an initial model B_ω, finite models B_n and a projective limit B^∞, which contains A^∞ as a substructure. Typically a process of the form

$$\partial_H(p||q) = r$$

with $p,q \in B^\infty$ will be in $(A \cup I)^\infty$ where $I \subseteq B - A$ is a set of so called *internal actions*.

In order to obtain a process r^* in A^∞ which is equivalent to r an *abstraction mechanism* is required which allows to conclude that

$$r \xleftrightarrow{}_I r^*$$

(r is equivalent to r^* modulo internal steps I). Such a mechanism is described in [5] and inspired by MILNER [9].

An equation $X = \tau(X_1,\ldots,X_k)$ over A^∞ is guarded if each X_i in τ is preceded in τ by some atomic action.

A system of *guarded* fixed point equations

$$X_i = \tau_i(X_i,\ldots,X_k) \quad i = 1,\ldots,k$$

always has a unique solution $\underline{X}_1,\ldots,\underline{X}_k$ in $(A^\infty)^k$.

P is called *recursively definable* if there exists a (finite) system of guarded fixed point equations with solutions $\underline{X}_1,\ldots,\underline{X}_k$ such that $\underline{X}_1 = p$.

Recursive definitions are the most appropriate specification method in process algebra.

2. TWO SPECIFICATIONS OF QUEUE

D is a finite set of data values, $\underline{D}$ denotes $\{\underline{d} \mid d \in D\}$, where $\underline{d}$ is a disjoint copy of d.

Consider the following equational specification for an algebra $(SEQ,*,\emptyset,d\in D) = I(\Sigma,E)$:

Σ: SORTS SEQ
FUNCTIONS: SEQ × SEQ → SEQ
CONSTANTS: $\emptyset$, $d \in D$.

E: $\emptyset * X = X * \emptyset = X$
$(X*Y)*Z = X*(Y*Z)$

According to [6] (Σ,E) is extended to a transition system specification, with actions A and transitions T as follows:

A: $D \cup \underline{D}$

T: $d\colon X \to d*X$ (for $d \in D$)
$\underline{d}\colon X*d \to X$ (for $d \in D$).

(Σ,E,A,T) is an algebraic specification of a state transition system in the sense of [6]. $I(\Sigma,E)$ denotes the initial algebra of (Σ,E).

We define a state of the system to be a point in $I(\Sigma,E)$ from which at least one transition is possible. Clearly in the present case all points in $I(\Sigma,E)$ are states. According to [6] one assigns to each state s of $I(\Sigma,E)$ a process $\pi(s)$ in A^∞. This step is called *behavioural abstraction*.

Rather than giving a formal definition of $\pi(s)$ in general we describe the properties of the $\pi(s)$ in our particular case:

$$\begin{cases} \pi(\emptyset) = \sum\limits_{d\in D} d.\pi(d) \\ \pi(s*d) = \underline{d}.\pi(s) + \sum\limits_{e\in D} e.\pi(e*s*d). \end{cases}$$

This is an infinite system of guarded equations which uniquely determine each $\pi(s) \in A^\infty$.

$\pi(\emptyset)$ corresponds to Q with empty initial state. This is the first specification of Q. The second specification aims at a finite system of fixed point equations for Q. We will prove that Q is the unique solution of the equation

$$Q = \sum_{d \in D} d.(Q \bigwedge \underline{d}).$$

Here for each $d \in D$, $\bigwedge \underline{d}$ is an auxiliary operator: $A^\infty \to A^\infty$, called Q-merge, which was introduced in [4].

The operators $\bigwedge \underline{d}$ and additional auxiliary operators $\triangle \underline{d}$ are simultaneously defined by means of these equations, (again a ranges over the atomic actions).

$X \bigwedge \underline{d} = \underline{d}.X + X \triangle \underline{d}$	QM1
$a \triangle \underline{d} = a.\underline{d}$	QM2
$\underline{b} \triangle \underline{d} = \delta$	QM3
$a.X \triangle \underline{d} = a(X \bigwedge \underline{d})$	QM4
$\underline{a}.X \triangle \underline{d} = \delta$	QM5
$(X+Y) \triangle \underline{d} = X \triangle \underline{d} + Y \triangle \underline{d}$	QM6

Here one works in $(A \cup \{\delta\})^\infty$ with the following axioms for δ:

$$X + \delta = X$$
$$\delta \,.\, X = \delta$$

which are axioms A6 and A7 of ACP.

In order to show that QM1 - 6 serve as a proper definition of $\bigwedge \underline{d}$ and $\triangle \underline{d}$ and do not introduce unwanted identifications on $(A \cup \{\delta\})_\omega$ or $(A \cup \{\delta\})^\infty$ a prooftheoretic analysis of QM1 - 6 is required. We will not perform this analysis here as it is essentially straightforward. Its results can be concisely summarised as follows:

PROPOSITION.

(i) $A_\omega(+,\cdot,\|,\mathbin{\|\!\_},\bigwedge\underline{d},\triangle\underline{d},\delta)$, *the initial algebra of* A1-7, M1-4, QM1-6 *is an enrichment of* A1-7, M1-4.

(ii) $X \equiv_n Y$ *implies* $X \bigwedge \underline{d} \equiv_n Y \bigwedge \underline{d}$ *and* $X \triangle \underline{d} \equiv_n Y \triangle \underline{d}$. *This guarantees that* $\bigwedge \underline{d}$ *and* $\triangle \underline{d}$ *can be extended to* $A^\infty(+,\cdot,\|,\mathbin{\|\!\_},\delta)$.

We will now establish this theorem:

THEOREM. $\pi(\emptyset) = \sum_{d \in D} d.(\pi(\emptyset) \wedge \underline{d})$.

From the the theorem it follows that the equation

$$X = \sum_{d \in D} d.(X \wedge \underline{d})$$

specifies $Q = \pi(\emptyset)$ because it has exactly one solution.

PROOF. The proof rests on a more general lemma:

LEMMA. *For all* s

$$\pi(s * \underline{d}) = \pi(s) \wedge \underline{d}.$$

Given this lemma the equation follows immediately:

$$\pi(\emptyset) = \sum_{d \in D} d.\pi(\emptyset * d) = \sum_{d \in D} d.(\pi(\emptyset) \wedge \underline{d}).$$

PROOF of the lemma. By induction on n we show that for all s:

$$(\pi(s*d))_n = (\pi(s) \wedge \underline{d})_n,$$

from which the required identity follows by definition. For each n there are two cases $s = \emptyset$ and $s = s' * e$.

(n=1) $s = \emptyset$:

$$(\pi(d))_1 = (\sum_{a \in D} a.\pi(a*d) + \underline{d}.\pi(\emptyset))_1 = \sum_{a \in D} a + \underline{d}$$

$$(\pi(\emptyset) \wedge \underline{d})_1 = (\underline{d}.\pi(\emptyset))_1 + ((\sum_{a \in D} a.\pi(a)) \wedge \underline{d})_1 =$$

$$\underline{d} + (\sum_{a \in D} a.(\pi(a) \wedge \underline{d}))_1 = \underline{d} + \sum_{a \in D} a.$$

$s = s'.e$: similar.

(n=m+1) $s = s' * e$:

$$(\pi(s*d))_n = (\pi(s'*e*d))_n =$$

$$\sum_{a\in D} a.(\pi(a*s'*e*d))_m + \underline{d}(\pi(s'*e))_m$$

$$(\pi(s)\triangle\underline{d})_n = (\pi(s'*e)\triangle\underline{d})_n =$$

$$\underline{d}.(\pi(s'*e))_m + (\pi(s'*e)\triangle\underline{d})_n .$$

Consider the second summand:

$$(\pi(s'*e)\triangle\underline{d})_n = ((\sum_{a\in D} a.\pi(a*s'*e) + \underline{e}.\pi(s'))\triangle\underline{d})_n =$$

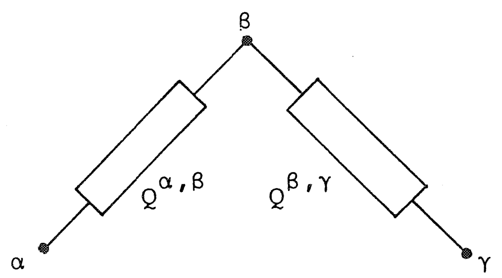
$$\sum_{a\in D} a.(\pi(a*s'*e)\triangle\underline{d})_m + \delta = \sum_{a\in D} a.(\pi(a*s'*e*d))_m .$$

This proves the identity for the case $s = s'*e$. The case $s = \emptyset$ is similar but shorter. □

This concludes the proof of the theorem, and makes a definition of Q within PA_δ available.

3. CONNECTING TWO QUEUES IN SERIES YIELDS ANOTHER QUEUE

Suppose there are three locations α,β and γ and a finite set of data D. Pairwise both α and β, and β and γ are connected with a queue that transmits values from D.



Actions of $Q^{\alpha,\beta}$ are d^{α}: input d at α and d^{β}: output d at β for $d \in D$.
Actions of $Q^{\beta,\gamma}$ are d^{β}: input d at β and d^{γ}: output d at γ.

In previous sections we have described the queue as a process Q in $(D \cup \underline{D})^{\infty}$.

Define $\phi^{\alpha,\beta}$: $D \cup \underline{D} \to D^{\alpha} \cup D^{\beta}$ by

$$\phi^{\alpha,\beta}(d) = d^{\alpha}$$
$$\phi^{\alpha,\beta}(\underline{d}) = d^{\beta}$$

and $\phi^{\beta,\gamma}$: $D \cup \underline{D} \to D^{\beta} \cup D^{\gamma}$ by

$$\phi^{\beta,\gamma}(d) = d^{\beta}$$
$$\phi^{\beta,\gamma}(\underline{d}) = d^{\gamma}.$$

The mapping of $\phi^{\alpha,\beta}$ induces an isomorphism

$$\phi^{\alpha,\beta}: (D \cup \underline{D})^{\infty} \to (D^{\alpha} \cup D^{\beta})^{\infty}.$$

A formal description of $Q^{\alpha,\beta}$ is then given by

$$Q^{\alpha,\beta} = \phi^{\alpha,\beta}(Q).$$

Similarly

$$Q^{\beta,\gamma} = \phi^{\beta,\gamma}(Q).$$

The problem is then to manufacture a queue $Q^{\alpha,\gamma}$ connecting α and γ by composing $Q^{\alpha,\beta}$ and $Q^{\beta,\gamma}$. Semantically $Q^{\alpha,\gamma}$ is given by

$$Q^{\alpha,\gamma} = \phi^{\alpha,\gamma}(Q) \in (D^{\alpha} \cup D^{\gamma})^{\infty}$$

with $\phi^{\alpha,\gamma}(d) = d^{\alpha}, \phi^{\alpha,\gamma}(\underline{d}) = d^{\gamma}$.

In order to compose both queues $Q^{\alpha,\beta}$ and $Q^{\beta,\gamma}$ there is communication at β. The actions d^{β} (from $Q^{\alpha,\beta}$) and d^{β} (from $Q^{\beta,\gamma}$) must be simultaneously performed:

$$d^{\beta} \mid d^{\beta} = \hat{d}^{\beta}.$$

$\hat{d}^{\beta}$ stands for: take d from $Q^{\alpha,\beta}$ and put it in $Q^{\beta,\gamma}$. There are no other communicating pairs of atomic actions than these, i.e., for all other pairs $\langle X,Y \rangle$ of actions, $X|Y = \delta$.
Technically this is described in ACP, an extension of PA (see [3]).

We work with alphabet A:

$$A = D^{\alpha} \cup D^{\beta} \cup D^{\gamma} \cup \{\hat{d}^{\beta} \mid d \in D\} \cup \{\delta\}.$$

Communication on atomic actions is in our case as follows:

$$\begin{cases} d^{\beta} \mid d^{\beta} = \hat{d}^{\beta} \text{ for } d \in D \\ a \mid b = \delta \text{ in all other cases.} \end{cases}$$

Moreover, $H \subseteq A$, the set of subatomic actions consists of the d^{β} for $d \in D$, and I, the set of internal actions, consists of the $\hat{d}^{\beta}$, $d \in D$.

The composition of $Q^{\alpha,\beta}$ and $Q^{\beta,\gamma}$ is now given by

$$R = \partial_H(Q^{\alpha,\beta} \parallel Q^{\beta,\gamma}).$$

This is a process in $(D^{\alpha} \cup \hat{D}^{\beta} \cup D^{\gamma})^{\infty}$. We write B for $D^{\alpha} \cup \hat{D}^{\beta} \cup D^{\gamma}$, I for $\hat{D}^{\beta}$ and E for $D^{\alpha} \cup D^{\gamma}$.

The relation between R and $Q^{\alpha,\gamma}$ is as follows:

$$R \underline{\leftrightarrow}_I Q^{\alpha,\gamma}.$$

This means that R and $Q^{\alpha,\gamma}$ bisimulate one another in the sense of [5]. The verification of

$$Q^{\alpha,\gamma} \underline{\leftrightarrow}_I \partial_H(Q^{\alpha,\beta} \parallel Q^{\beta,\gamma}) \quad (=R)$$

should be considered a correctness proof of the realisation of $Q^{\alpha,\gamma}$ via $Q^{\alpha,\beta}$ and $Q^{\beta,\gamma}$.

The verification takes place within B^{∞}, and consists of proving that for each n

$$PAI \vdash (Q^{\alpha,\gamma})^e_n = (R)^e_n.$$

Here $(X)^e_n$ is the n-th external projection of X, inductively defined by

$$\begin{cases} (aX)^e_1 = a \quad \text{for } a \in E \\ (aX)^e_{n+1} = a(X)^e_n \quad \text{for } a \in E \\ (X+Y)^e_{n+1} = (X)^e_n + (Y)^e_n \\ (iX)^e_n = i(X^e_n) \\ (i)^e_n = i \end{cases}$$

PAI is PA augmented with the following equations:

$i = j$	I1
$Xi = X$	I2
$iX + X = iX$	I3
$a(iX+Y) + aX = a(iX+Y)$	I4

where i,j range over $I \subseteq B$. PAI was introduced and investigated in [5]. The laws correspond to Milner's τ-laws, from [9]. For the sake of abbreviation we write:

$$\partial_H(\phi^{\alpha,\beta}(\pi(\sigma)) \,||\, \phi^{\beta,\gamma}(\pi(\tau))) = \pi(\sigma,\tau)$$

and $\phi^{\alpha,\gamma}(\pi(\sigma)) = \pi^{\alpha,\gamma}(\sigma)$. We will now establish several propositions.

PROPOSITION. *The following identities hold for the* $\pi(\sigma,\tau)$:

(i) $\pi(\emptyset,\emptyset) = \Sigma_{d\in D}\, d^{\alpha}.\pi(d,\emptyset)$

(ii) $\pi(\sigma * d,\emptyset) = \Sigma_{b\in D}\, b^{\alpha}.\pi(b*\sigma*d,\emptyset) + \hat{d}^{\beta}.\pi(\sigma,d)$

(iii) $\pi(\sigma*d,\tau*e) = \Sigma_{b\in D}\, b^{\alpha}.\pi(b*\sigma*d,\tau*e) + \hat{d}^{\beta}.\pi(\sigma,d*\tau*e) + e^{\gamma}.\pi(\sigma*d,\tau)$

(iv) $\pi(\emptyset,\tau*e) = \Sigma_{b\in D}\, b^{\alpha}.\pi(b,\tau*e) + e^{\gamma}\pi(\emptyset,\tau)$.

PROOF. All identities follow from straightforward calculations in ACP. We take one example:

$$\pi(\sigma*d,\emptyset) = \partial_H(\phi^{\alpha,\beta}(\pi(\sigma*d))) \,||\, \phi^{\beta,\gamma}(\pi(\emptyset))) =$$

$$\partial_H(\phi^{\alpha,\beta}(\pi(\sigma*d)) \,\lfloor\!\lfloor\, \phi^{\beta,\gamma}(\pi(\emptyset))) + \qquad (T_1)$$

$$\partial_H(\phi^{\beta,\gamma}(\pi(\emptyset)) \,\lfloor\!\lfloor\, \phi^{\alpha,\beta}(\pi(\sigma*d)) + \qquad (T_2)$$

$$\partial_H(\phi^{\alpha,\beta}(\pi(\sigma*d)) \,|\, \phi^{\beta,\gamma}(\pi(\emptyset)). \qquad (T_3)$$

Now

$$T_1 = \partial_H(\phi^{\alpha,\beta}(\Sigma_{b\in D}\ b.\pi(\sigma * d) + \underline{d}\pi(\sigma)) \lfloor\!\lfloor \phi^{\beta,\gamma}(\pi(\emptyset)))$$

$$= {}_H((\Sigma_{b\in D}\ b^{\alpha}.\phi^{\alpha,\beta}(\pi(b*\sigma*d)) + d^{\beta}.\phi^{\alpha,\beta}(\pi(\sigma))) \lfloor\!\lfloor \phi^{\beta,\gamma}(\pi(\emptyset)))$$

$$= \partial_H(\Sigma_{b\in D}\ b^{\alpha}(\phi^{\alpha,\beta}\pi(b*\sigma*d) \parallel \phi^{\beta,\gamma}(\pi(\emptyset))$$

$$+ \partial_H d^{\beta}.(\phi^{\alpha,\beta}\pi(\sigma) \parallel \phi^{\beta,\gamma}(\pi(\emptyset)))$$

$$= \Sigma_{\sigma\in D}\ b^{\alpha}.\pi(b*\sigma*d,\emptyset).$$

$$T_2 = \partial_H(\phi^{\beta,\gamma}(\Sigma_{b\in D}\ b.\pi(b)) \lfloor\!\lfloor \phi^{\alpha,\beta}(\pi(\sigma*d)))$$

$$= \partial_H\ \Sigma_{b\in D}\ b^{\beta}(\phi^{\beta,\gamma} \parallel \phi^{\alpha,\beta}(\pi(\sigma*d))) = \delta.$$

$$T_3 = \partial_H(\phi^{\alpha,\beta}(\Sigma_{b\in b}\ b.\pi(b*\sigma*d) + \underline{d}\pi(\sigma)) \mid \phi^{\beta,\gamma}(\Sigma_{e\in D}\ e.\pi(e)))$$

$$= \partial_H\ \Sigma_{b\in D}\ \Sigma_{e\in D}\ (b^{\alpha} \mid e^{\beta})(\phi^{\alpha,\beta}(\pi(b*\sigma*d) \parallel \phi^{\beta,\gamma}(\pi(e)))$$

$$+ \partial_H\ \Sigma_{e\in D}\ (d^{\beta} \mid e^{\beta})(\phi^{\alpha,\beta}(\pi(\sigma)) \parallel \phi^{\beta,\gamma}(\pi(e)))$$

$$= \hat{d}^{\beta}.\partial_H(\phi^{\alpha,\beta}(\pi(\sigma)) \parallel \phi^{\beta,\gamma}(\pi(d)))$$

$$= \hat{d}^{\beta}.\pi(\sigma,d).$$

We conclude that $T_1 + T_2 + T_3 = \Sigma_{b\in D}\ b^{\alpha}.\pi(b*\sigma*d,\phi) + \hat{d}^{\beta}\pi(\sigma,d)$. This proves the equation (ii) for $\pi(\cdot,\cdot)$. The other identities are established similarly.

<u>PROPOSITION</u>. *For all* σ,τ,n

$$PAI \vdash (\pi(\sigma,\tau))^{e}_{n} = i(\pi(\phi,\sigma*\tau))^{e}_{n}.$$

<u>PROOF</u>. The proof proceeds by double induction on n and on the length ℓ of σ. i stands for any fixed element of I.

(n=1): the case $\ell=0$ is immediate, so let $\ell = \ell'+1$, and write $\sigma = \sigma'*d$. There are two cases: $1th(\tau) = 0$ and $1th(\tau) > 0$, we consider the second case only, let $\tau = \tau'*b$, then

$$(\pi(\sigma,\tau))_1^e = (\pi(\sigma'*d,\tau'*b))_1^e =$$

$$\Sigma_{e\in D}\ e^\alpha + \hat{d}^\beta(\pi(\sigma',d*\tau'*b))_1^e + b^\gamma$$

= (induction hypothesis on ℓ)

$$\Sigma_{e\in D}\ e^\alpha + i.i(\pi(\emptyset,\sigma'*d*\tau'*b))_1^e + b^\gamma =$$

$$\Sigma_{e\in D}\ e^\alpha + i(\Sigma_{e\in D}\ e^\alpha + b^\gamma) + b^\gamma =$$

$$i(\Sigma_{e\in D}\ e^\alpha + b^\gamma)$$

(because iX + X = iX).

(n=m+1): again the case $\ell = 0$ is immediate. So suppose $\ell = \ell'+1$ $\sigma = \sigma'*d$. Again there are two cases $1th(\tau) = 0$ and $1th(\tau) > 0$. This time we consider the case $1th(\tau) = 0$, (the other case leads to a similar calculation)

$$(\pi(\sigma,\tau))_n^e = (\pi(\sigma'*d,\emptyset))_{m+1}^e =$$

$$\underset{(1)}{=} \Sigma_{e\in D}\ e^\alpha(\pi(e*\sigma'*d,\emptyset))_m^e + \hat{d}^\beta(\pi(\sigma',d))_n^e =$$

$$\underset{(2)}{=} \Sigma_{e\in D}\ e^\alpha(\pi(\emptyset,e*\sigma'*d))_m^e + i[\Sigma_{e\in D}\ e^\alpha(\pi(e*\sigma',d))_m^e + (\pi(\sigma',d))_n^e]$$

$$\underset{(3)}{=} i[\Sigma_{e\in D}\ e^\alpha(\pi(\emptyset,e*\sigma'*d))_m^e + (\pi(\sigma',d))_n^e] =$$

$$\underset{(4)}{=} i(\pi(\sigma',d))_n^e \underset{(5)}{=} i(\pi(\emptyset,\sigma'*d))_n^e.$$

Comments.

(1) by equations for $\pi(\cdot,\cdot)$.

(2) ind. hyp. on m, equations for $\pi(\cdot,\cdot)$.

(3) according to i(X+Y) + X = i(X+Y), a derived rule of PAI.

(4) by equations for $\pi(\cdot,\cdot)$

(5) ind. hyp. on ℓ.

PROPOSITION. *For all* σ,n

$$PAI \vdash (\pi^{\alpha,\gamma}(\sigma))_n^e = (\pi(\emptyset,\sigma))_n^e.$$

PROOF. Induction on n

$(n=1)\ \sigma = \emptyset$:

$$(\pi^{\alpha,\gamma}(\emptyset))^e_1 = \Sigma_{d\in D}\ d^\alpha = (\pi(\emptyset,\emptyset))^e_1$$

$\sigma = \sigma' * d$:

$$(\pi^{\alpha,\gamma}(\sigma' * d))^e_1 = \Sigma_{e\in D}\ e^\alpha + d^\gamma = (\pi(\emptyset,\sigma' * d))^e_1$$

$(n=m+1)\ \sigma = \emptyset$:

$$(\pi^{\alpha,\gamma}(\emptyset))^e_n = \Sigma_{d\in D}\ d^\alpha.(\pi(d))^e_m \overset{(1)}{=} \Sigma_{d\in D}\ d^\alpha.(\pi(\emptyset,d))^e_m \overset{(1)}{=}$$

$$\Sigma_{d\in D}\ d^\alpha.i.(\pi(d,\emptyset))^e_m \overset{(3)}{=} \Sigma_{d\in D}\ d^\alpha(\pi(d,\emptyset))^e_m \overset{(4)}{=}$$

$$(\pi(\emptyset,\emptyset))^e_n.$$

Comments.

(1) ind. hypothesis.

(2) application of the previous proposition.

(3) $Xi = X$ from PAI.

(4) equation for $\pi(\emptyset,\emptyset)$.

The case $\sigma = \sigma' * d$ is similar.

Finally the required result follows from

$$\text{PAI} \vdash (\pi^{\alpha,\gamma}(\emptyset))^e_n = (\pi(\emptyset,\emptyset))^e_n$$

i.e. $$\text{PAI} \vdash (Q^{\alpha,\gamma})^e_n = (R)^e_n.$$

This completes the verification of

$$Q^{\alpha,\gamma} \underline{\leftrightarrow}_I R.$$

4. QUEUE CANNOT BE RECURSIVELY DEFINED IN $A^\infty(+,\cdot,\|,\lfloor\!\lfloor)$

In this section we are going to prove the above statement. We assume that A has two different input actions a and b. A queue over a one-element set of input actions is just a "bag" and it is easily definable in $A^{\infty}(+,\cdot,\|,\lfloor\!\lfloor)$ (see [4]).

This section is organized as follows. We start with preliminary definitions in section 4.1, and prove some auxiliary results in section 4.2. In section 4.3 the problem in first reduced to the same problem without $\|$ and $\lfloor\!\lfloor$. Then the latter question is settled in the negative.

4.1 Preliminaries

4.1.1 DEFINITION

Let $p \in A^{\infty}$. By the set of *states* of p we mean the least set ST(p) satisfying the following conditions

(1) $p \in ST(p)$

(2) If $c.q + r \in ST(p)$, then $q \in ST(p)$, for $c \in A$, $q,r \in A^{\infty}$.

4.1.2 DEFINITION

The set of all *semistates* of p, SST(p), is the least set satisfying these conditions

(3) $ST(p) \subseteq SST(p)$

(4) If $q + r \in SST(p)$, then $q \in SST(p)$ for $q,r \in A^{\infty}$.

4.1.3 DEFINITION

We say that a process h is a *factor* of a process p if for some $q \in A^{\infty}$, $h.q = p$.

In the context of this proof, however, a factor will be any process which is a factor of a semistate of Q. Let F(Q) denote the set of all factors.

A *trivial factor* is a factor in $F(Q) \cap A$. All other factors will be called *nontrivial*.

The following inclusions are obvious.

$$ST(Q) \subseteq SST(Q) \subseteq F(Q).$$

4.1.4 Let $\sigma \in A^{*}$, $p,q \in A^{\infty}$. We are going to define the relation $\sigma: p \to q$ which intuitively means that σ is a *path* in p which leads to q.

If $\phi \in A^*$ is the empty word, then ϕ: $p \to q$ iff there exists $r \in A^\infty$ such that $p = q+r$.

Let $c \in A$, c: $p \to q$ iff $p = cq+r$ for some $q,r \in A^\infty$.

Finally, σc: $p \to q$ iff for some $r \in A^\infty$ σ: $p \to r$, and c: $r \to q$.
The following fact is easy to prove by induction on the length of σ.

4.1.5 Fact

(i) For every $\sigma \in A^*$ and for every $p,q \in A^\infty$, if σ: $p \to q$, then $q \in SST(p)$.
(ii) Moreover, for every $q \in SST(p)$ there exists $\sigma \in A^*$ such that σ: $p \to q$.

In the case of Q we can deduce more.

4.1.6 Fact

(i) For every $q \in SST(Q)$ there exists a unique $\sigma \in \{a,b\}^*$ such that σ: $Q \to q$.
(ii) For every $q \in ST(Q)$ and for every $\sigma \in A^*$, σ: $Q \to q$ iff $\pi(\sigma) = q$.

The proof of this fact is easy and we leave it for the reader.

4.1.7 Now we are going to define the notion of a trace of a process $p \in A^\infty$.

A finite word $\sigma \in A^*$ is a trace if there exist $c \in A$ and $\tau \in A^*$ such that

$$\sigma = \tau c$$

and

$$\tau: p \to c.$$

An infinite word $V \in A^\omega$ is a trace of p if there exists a sequence of initial fragments of V,

$$\sigma_1 < \sigma_2 < \dots < \sigma_n < \dots$$

and a sequence of semistates of p,

$$q_1, q_2, \dots, q_n, \dots$$

such that for every $n \in \omega$

$$\sigma_n: p \to q_n \quad \text{and} \quad q_{n+1} \in SST(q_n).$$

Let tr(p) denote the set containing all finite and infinite traces of p.

A process is called *perpetual* iff it contains no finite traces. The following fact is obvious.

4.1.8 Fact

Let $\sigma \in tr(p) \cap A^{*}$. Then for every $q \in A^{\infty}$, $\sigma: pq \to q$.
Another general property of A^{∞} which we will use later has a straightforward proof as well.

4.1.9 Fact

For all $q,r,p \in A^{\infty}$ and $c \in A$, if $q + r = cp$, then $q = r$.

4.1.10 We introduce two functions

$$O,I: A^{\omega} \to \{a,b\}^{*} \cup \{a,b\}^{\omega}.$$

They are uniquely determined by the following properties

If $V \in \{a,b\}^{\omega}$, then $O(V) = \emptyset$
If $V \in \{\underline{a},\underline{b}\}^{\omega}$, then $I(V) = \emptyset$
If $c \in \{a,b\}$ and $V \in A^{\omega}$, then

$$I(cV) = cI(V), \text{ and } O(CV) = O(V).$$

If $\underline{c} \in \{\underline{a},\underline{b}\}$ and $V \in A^{\omega}$, then

$$I(\underline{c}V) = I(V), \text{ and } O(\underline{c}V) = cO(V).$$

Intuitively $I(V)$ (respectively $O(V)$) is the sequence of all input (output) actions of V in the order in which they occur in V.

Call $V \in A^{\omega}$ *input periodic* if there exists $\sigma \in \{a,b\}^{+}$ such that $I(V) = \sigma^{\omega}$. Otherwise V will be called input nonperiodic.

$V \in A^{\omega}$ is said to have infinitely many output actions if $O(V) \in \{a,b\}^{\omega}$.

The last result of this subsection is the following lemma.

4.1.11 LEMMA *Let* $p_0,p_1 \in SST(Q)$ *be such that* $tr(p_0) \cap tr(p_1)$ *contains an input nonperiodic trace with infinitely many output actions. Then for some* $\sigma \in \{a,b\}^{*}$, $\sigma: Q \to p_0$ *and* $\sigma: Q \to p_1$.

PROOF. Let $\sigma_0: Q \to p_0$ and $\sigma_1: Q \to p_1$ for some $\sigma_0, \sigma_1 \in \{a,b\}^*$. Assume that $\sigma_0 \neq \sigma_1$. Let $V \in tr(p_0) \cap tr(p_1)$ be a trace with infinitely many output actions. We are going to show that V is input periodic.

Since $\sigma_i: Q \to p_i$, it follows that $\sigma_i \leq O(V)$ for $i = 0,1$. Therefore either $\sigma_0 < \sigma_1$ or $\sigma_1 < \sigma_0$. We may asssume without loss of generality that $\sigma_0 < \sigma_1$. Let $\tau \in \{a,b\}^+$ be such that

$$\sigma_1 = \sigma_0 . \tau.$$

Let V_0 be a sequence which results from V by removing from it the first $|\sigma_0|$ output actions. Since $V \in tr(p_0)$, it follows that $V_0 \in tr(Q)$.

Let V_1 be a sequence which results from V by removing from it the first $|\sigma_1|$ output actions. Again we have $V_1 \in tr(Q)$. Moreover we have the following relationships

(1) $I(V_0) = I(V_1) = I(V)$,

(2) $O(V) = \sigma_0 \, O(V_0)$,

(3) $O(V) = \sigma_0 \tau \, O(V_1)$.

From (2) and (3) we obtain

(4) $O(V_0) = \tau \, O(V_1)$.

We claim that

(5) for every $n \in \omega$ there exist $W_n, U_n \in A^\omega$ such that

$$I(V_0) = \tau^n W_n \text{ and } O(V_1) = \tau^n U_n.$$

We prove this claim by induction on n. For $n = 0$ it is obvious. Suppose (5) holds for some $n \in \omega$. By (4) we have

(6) $O(V_0) = \tau^{n+1} U_n$.

Since V_0 is a trace of Q with infinitely many output actions, it follows that $O(V_0) = I(V_0)$. Therefore for some $W_{n+1} \in A^\omega$,

(7) $I(V_0) = \tau^{n+1} W_{n+1}$.

Since V_1 is a trace of Q, by (1) and (7) we conclude that $O(V_1)$ must start with τ^{n+1}, i.e. for some $U_{n+1} \in A^\omega$,

(8) $O(V_1) = \tau^{n+1} U_{n+1}$.

This proves (5). Obviously (5) implies that

$$I(V_0) = \tau^\omega.$$

This, together with (1) proves the lemma.

4.2 Auxiliary results

Let us start with the following result.

4.2.1 LEMMA *Let* h *be a nontrivial factor. Then*
(i) h *has an input-nonperiodic trace with infinitely many output actions.*
(ii) *there is a unique* $p \in SST(Q)$ *such that* h *is a factor of* p.

PROOF. We prove (i) first. Let h be a nontrivial factor and let $q \in A^{\infty}$ be such that $hq \in SST(Q)$. If h is perpetual, then $hq = h$ and h is a semistate of Q. Then h obviously satisfies (i).

Suppose now that h is not perpetual and let σ be a finite trace of h. By Fact 4.1.8, σ: hq $\to$ q. Therefore, by Fact 4.1.5 $q \in SST(Q)$. Let $\tau, \rho \in \{a,b\}^*$ be such that τ: Q $\to$ q, and ρ: Q $\to$ hq (cf. Fact 4.1.6).

Since hq is a semistate of Q it can be uniquely presented in the following form

(*) $$hq = \Sigma_{c \in C}\, c.p_c,$$

where $C \subseteq \{a,b,\underline{a},\underline{b}\}$, and $p_c \in A^{\infty}$ for every $c \in C$.
Consider these two cases
(1) There exists $c \in C$ such that *non* ρc: C $\to$ q.
(2) For all $c \in C$, ρc: Q $\to$ q.
Notice that (2) may happen only if C is a one element set.

Suppose (1) holds. We construct a $V \in A^{\omega}$ such that
(3) $V \in tr(\pi(\rho c))$ is input-nonperiodic
(4) for every initial seqment α of V, *non* $\rho c\alpha$: Q $\to$ q.
(5) V has infinitely many output actions.
To see that such V exists take $\rho' \in \{a,b\}^*$ such that $\pi(\rho') = \pi(\rho c)$. If ρ' is not an initial subword of τ, then we may take as V any trace of Q which is input-nonperiodic and which has infinitely many output actions. If, however, ρ' is an initial subword of τ, then it follows from (1) that $\rho' \neq \tau$. Then it is enough to take as V eW, where $e \in \{a,b\}$ is such that $\rho'e$ is not an initial subword of τ, and $W \in tr(Q)$ is any input-nonperiodic trace with infinitely many output actions.

Let V be a trace satisfying (3)-(5). By (*) and (3), $cV \in tr(hq)$, and by (4) we obtain that cV must be a trace of h.

Suppose now that (2) holds. As we noticed this may happen only if C is a one element set, say $C = \{c\}$. Since $hq = cp_c$ and since h is nontrivial, there exist $h_1, h_2 \in A^\infty$ such that

$$h = ch_1 + h_2.$$

Therefore,

$$ch_1q + h_2q = cp_c.$$

By Fact 4.1.9, $ch_1q = h_2q$, and we obtain

$$ch_1q = cp_c.$$

The latter equality can be simplified to

$$h_1q = p_c.$$

Thus h_1 is a factor of a state of Q, and case (1) is applicable to h_1. We may conclude that h_1 contains an input-nonperiodic trace with infinitely many output actions, and therefore h contains such a trace as well. This completes the proof of (i).

Now we prove (ii).

Let $p_0, p_1 \in SST(Q)$ be such that for some $q_0, q_1 \in A^\infty$

(6) $h.q_i = p_i$ for $i = 0,1$.

By (i) h has an input-nonperiodic trace with infintely many output actions. Therefore, by (6), $tr(p_0) \cap tr(p_1)$ contains such a trace and by Lemma 4.1.11 for some $\sigma \in \{a,b\}^*$,

(7) $\sigma\colon Q \to p_i$, for $i = 0,1$.

If $p_0 \neq p_1$, then by (7) this is only possible if for some $c \in A$ and $i_0 \in \{0,1\}$, p_{i_0} has a trace starting with c and in p_{1-i_0} all traces start with a symbol different from c. Since $hq_{i_0} = p_{i_0}$, h must have a trace starting with c. Since $hq_{1-i_0} = p_{1-i_0}$, p_{1-i_0} must have such a trace as well. Obtained contradiction proves (ii), and the proof of Lemma 4.2.1 is completed.

4.2.2 LEMMA

(i) *if* $h_1 + h_2 \in F(Q)$ *then* $h_1, h_2 \in F(Q)$

(ii) *if* $h_1.h_2 \in F(Q)$ *then* $h_1 \in F(Q)$, *moreover if* h_1 *is not perpetual then*

$h_2 \in F(Q)$ *as well*

(iii) $h_1 \| h_2 \notin F(Q)$

(iv) $h_1 \mathbin{⫫} h_2 \notin F(Q)$ *provided* h_1 *is not a sum of atoms.*

PROOF. (i) Suppose $(h_1+h_2).r = q$, $q \in SST(Q)$, then $h_1.r + h_2.r = q$ thus $h_1.r$, $h_2.r \in SST(Q)$ whence by definition $h_1, h_2 \in F(Q)$.

(ii) Suppose $(h_1.h_2).r = q$, $q \in SST(Q)$, then $h_1(h_2.r) = q$ so $h_1 \in F(Q)$. If σ is a finite trace of h_1 then by 4.1.8. σ: $h_1(h_2.r) \to h_2.r$, and also σ: $q \to h_2.r$. So $h_2.r \in ST(Q)$ by fact 4.1.5 and by definition $h_2 \in F(Q)$.

(iii) Suppose $h_1 \| h_2 \in F(Q)$. Let $q \in A^\infty$ be such that $(h_1 \| h_2).q \in SST(Q)$, $h_1 \| h_2$ cannot be $a, b, \underline{a}$, or $\underline{b}$ so $h_1 \| h_2$ is a nontrivial factor. In view of Lemma 4.2.1 $h_1 \| h_2$ has an input nonperiodic infinite trace V with infinitely many output actions. V must have both infinitely many $\underline{a}$'s and $\underline{b}$'s. Let V1 and V2 be traces of h1 resp. h2 such that V can be obtained by merging V1 and V2.

Choose $\sigma \in \{a,b\}^*$ such that σ: $Q \to (h1 \| h2).q$. Then for every sequence U obtained by merging V1 and V2 $\sigma U \in tr(Q)$. We will manufacture a contradiction from this situation. First of all we notice that either V1 or V2 contains no output actions, otherwise V1 must contain an action $\underline{a}$ and V2 an action $\underline{b}$ or conversely. Let us assume that $V1 = \sigma_1 \underline{a}U1$, $V_2 = \sigma_2\underline{b}U2$ then $\sigma\sigma_1\sigma_2\underline{a}U1$ and $\sigma\sigma_1\sigma_2\underline{b}U1$ are both traces of Q (because V1 is infinite U1 is infinite and both are σ followed by a merge of V1 and V2). Now this is impossible because after $\sigma\sigma_1\sigma_2$ at most one output is possible. So suppose that V1 contains infinitely many $\underline{a}$'s and $\underline{b}$'s and V2 contains only a's and b's; $\sigma V1$ is a trace of Q, inserting the first action, say a, from V2 in $\sigma V1$ at some position after σ must also produce a trace of Q. However choose ρ, W such that $V1 = \rho bW$ then $\sigma\rho abW$ cannot be a trace of Q because the output action in V_2^* that corresponds to the displayed input b has now become incorrect. Thus we have obtained a contradiction thereby proving (iii) of the Lemma.

(iv) the case for ⫫ is similar to the previous one.

Let us now consider a recursive definition of Q:

$$X_i = T_i(X_1,\ldots,X_n) \quad i = 1,\ldots,n,$$

with solutions $\underline{X}_1,\ldots,X_n$, and $\underline{X}_1 = Q$.

Without loss of generality we may assume that the system has the following properties:

(i) All $\underline{X}_i$ are infinite (otherwise they can be eliminated by substitution).

(ii) All $\underline{X}_i$ are "used" (i.e. no proper subsystem defines $\underline{X}_1 = Q$ as well).

(iii) None of the T_i has a subterm of the form $(t_2+t_2).t_3$. (Such subterms are eliminated using the equation A4.)

(iv) None of the T_i has a subterm t_1t_2 with $t_1(\underline{X}_1,\ldots,\underline{X}_n)$ perpetual (in such cases t_2 can just be omitted and the $\underline{X}_1,\ldots,\underline{X}_n$ still constitute a unique solution).

(v) In subterms of the form $t_1 \mathbin{\|\!\lfloor} t_2$, t_1 is not a sum of atoms. (Otherwise use CM4 and CM3).

We can now start the actual proof of the result of section 4 in the following lemma.

4.3.1 LEMMA. *If* Q *can be recursively defined in* $A^\infty(+,\cdot,\|,\mathbin{\|\!\lfloor})$ *then it can be recursively defined in* $A^\infty(+,\cdot)$.

PROOF. We assume that the system

$$X_i = T_i(X_1,\ldots,X_n) \quad i = 1,\ldots,n$$

satisfies the requirements (i)-(v) above and defines Q. Let Z be the collection of all subterms of the $T_i(X_1,\ldots,X_n)$. We can define a distance $d(\cdot,\cdot)$ between elements of Z. d is not symmetric, however:

(i) $d(t,t) = 0$

(ii) if t' is an immediate subterm of t then $d(t,t') = 1$

(iii) $d(t_1,t_2) = \min\{d(t_1,t')+d(t',t_2) \mid t' \in K\}$.

Now it follows that for each $t \in K$ $d(X,t)$ is defined. With induction on $d(X,t)$ one shows using Lemma 4.2.2 that each $t \in K$ is in F(Q). Moreover by 4.2.2 we conclude that $\|$ and $\mathbin{\|\!\lfloor}$ do not occur in any of the $t \in K$. This proves the lemma.

4.3.2 LEMMA. *If* Q *has a recursive definition in* $A^\infty(+,\cdot)$ *then* ST(Q) *is generated (in* $A^\infty(+,\cdot)$*) by finitely many states* $\pi(\sigma_1),\ldots,\pi(\sigma_K) \in ST(Q)$.

PROOF. According to [7] it is ingeneral the case that the solutions $\underline{X}_1,\ldots,X_n$ of a system of recursion equations generate all subprocesses of $\underline{X}_1,\ldots,\underline{X}_n$

and in particular of $\underline{X}_1$.

So let $T_i(X_i,\ldots,X_n)$, $i = 1,\ldots,n$, be a recursive definition in $A^\infty(+,\cdot)$ with solution $\underline{X}_1,\ldots,\underline{X}_n$ and $\underline{X}_1 = Q$, again satisfying (i)-(v) above, then for each $q \in ST(Q)$ there is a term $t(X_1,\ldots,X_n)$ made from A,+, and. such that $t(\underline{X}_1,\ldots,\underline{X}_n) = q$.

Let ϕ: $F(Q) \to SST(Q)$ be the mapping which assigns to each $h \in F(Q)$ the unique (in view of Lemma 4.2.1 (ii)) $q \in SST(Q)$ such that for some $p \in A^\infty$ $h.p = q$.

CLAIM. $\phi t(\underline{X}_1,\ldots,\underline{X}_n) = t(\phi(\underline{X}),\ldots,\phi(\underline{X}_n))$.

Using this claim one finds that $p = \phi(p)$ is generated by the semistates $\phi(\underline{X}_1),\ldots, (\underline{X}_n)$. Now each semistate $\phi(\underline{X}_i)$ can be written as

$$C^i_1\pi(\sigma^i_1) + C^i_2\pi(\sigma^i_2) + C^i_3\pi(\sigma^i_3),$$

for appropriate $C^i_j \in A$ and $\sigma^i_j \in \{a,b\}^*$.
It follows that the subset

$$\{\pi(\sigma^i_j) \mid 1 \leq j \leq n,\ 1 \leq i \leq 3\}$$

of ST(Q) generates all of ST(Q), thus proving the lemma.

PROOF (of the Claim). Let $L = L(X_1,\ldots,X_n)$ be the following inductively defined collection of terms:

(i) $X_i \in L$

(ii) $c.t \in L$ for $c \in A$, $t \in L$

(iii) $X_i.t \in L$ if $t \in L$

(iv) $t_1 + t_2 \in L$ if $t_1,t_2 \in L$

(v) $c \in L$ for $c \in A$.

Now each term t over $t,\cdot,A,X_1,\ldots,X_n$ in equivalent in PA to a term in L, and therefore it suffices to prove the claim for every term in L. With induction on the structure of $t \in L$ we will show this implication, which proves the claim:

$$t(\underline{X}_1,\ldots,\underline{X}_n) \in SST(Q) \Rightarrow \phi t(\underline{X}_1,\ldots,\underline{X}_n) = t(\phi\underline{X}_1,\ldots,\underline{X}_n).$$

We consider all cases generated by the inductive clauses (i),...,(v).

(i) is immediate

(ii) if $c.t(\underline{X}_1,\ldots,\underline{X}_n) \in SST(Q)$ then $t(\underline{X}_1,\ldots,\underline{X}_n) \in SST(Q)$. So $\phi(t(\underline{X}_1,\ldots,\underline{X}_n))= t(\phi(\underline{X}_1),\ldots,\phi(\underline{X}_n))$ and $\phi(c.t(\underline{X}_1,\ldots,\underline{X}_n)) = c.t(\phi(\underline{X}_1),\ldots,\phi(\underline{X}_n))$.

(iii) if $\phi(X_i.t(\underline{X}_1,\ldots,\underline{X}_n)) \in SST(Q)$ then $\phi(\underline{X}_i.t(\underline{X}_1,\ldots,\underline{X}_n)) = \underline{X}_i.t(\underline{X}_1,\ldots,\underline{X}_n) = \phi(\underline{X}_i) = \phi(\underline{X}_i).t(\phi(\underline{X}_1),\ldots,\phi(\underline{X}_n))$.

(iv) if $t_1(\underline{X}_1,\ldots,\underline{X}_n) + t_2(\underline{X}_1,\ldots,\underline{X}_n) \in SST(Q)$ then both summands are in SST(Q) hence $\phi(t_1(\underline{X}_1,\ldots,\underline{X}_n) + t_2(\underline{X}_1,\ldots,\underline{X}_n)) = t_1(\phi(\underline{X}_1),\ldots,\phi(\underline{X}_n)) + t_2(\phi(\underline{X}_1),\ldots,\phi(\underline{X}_n))$

(v) c is not in SST(Q).

4.3.3 LEMMA. *There is no finite subset* $\pi(\sigma_1)\ldots\pi(\sigma_K)$ *of* ST(Q) *which generates all of* ST(Q) *within* $A^\infty(+,\cdot)$.

PROOF. Suppose otherwise. Choose for each $\pi(\sigma_i)$ a triple $\pi(\tau_i^1)$, $\pi(\tau_i^2)$, $\pi(\tau_i^3)$ such that for appropriate $C_i^j \in A$

$$e_i\colon \pi(\sigma_i) = C_i^1.\pi(\tau_i^1) + C_i^2\pi(\tau_i^2) + C_i^3\pi(\tau_i^3),$$

then choose for each $\pi(\tau_i^j)$ a term $t_i^j(X_1,\ldots,X_K)$ such that

$$\pi(\tau_i^j) = t_i^j(\pi(\sigma_1),\ldots,\pi(\sigma_k)).$$

The term t_i^j may be chosen such that it contains + and prefix multiplication only because all $\pi(\sigma_i)$ are perpetual and $\pi(\sigma_i).t$ can be replaced by $\pi(\sigma_i)$.

Substituting these identities into e_i one obtains a linear system of equations for the processes $\pi(\sigma_i)$. According to [7] the $\pi(\sigma_i)$ will then be regular which is certainly not the case.

Combining lemmas 4.3.1, 4.3.2 and 4.3.3 we obtain the main result of this section:

THEOREM. Q *cannot be recursively defined in* $A^\infty(+,\cdot,\|,\lfloor\!\lfloor)$.

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